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of which the first is Wilson's Theorem, while the left member of the second is the coefficient of $x^m y^{p-1-m}$ in $(x+y)^{p-1}$. The latter is congruent, modulo p , to

$$\frac{x^p + y^p}{x + y} = x^{p-1} - x^{p-2}y + \dots + (-1)^m x^m y^{p-1-m} + \dots + y^{p-1}.$$

AVERAGE AND PROBABILITY.

130. Proposed by LON C. WALKER, A. M., Graduate Student, Leland Stanford University, Cal.

Four points are taken at random on the surface of a given sphere; show that the average volume of a tetrahedron formed by the planes passing through the points taken three and three, is $1/35$ of the volume of the given sphere.

I. Solution by the PROPOSER.

Choose A, B, C, D as the four random points; O the center of the given sphere with radius r : $ABFE$ a great circle through A, B ; ABC a small circle through A, B, C , with center S ; DGF a small circle through D parallel to ABC , with center P ; M the middle point of AB .

Put $OP=x$, $AS=r_1$, $\angle AOB=\theta$, $\angle OMS=\phi$, $\angle CAB=\psi$, $\angle SAM=\psi_1$. Then we have

$$AM = r \sin \frac{1}{2} \theta = r_1 \cos \psi_1,$$

$$SM = r \cos \frac{1}{2} \theta \cos \phi = r_1 \sin \psi_1,$$

$$OS = r \cos \frac{1}{2} \theta \sin \phi,$$

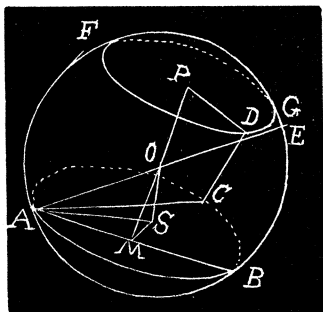
$$AC = 2r_1 \cos(\psi - \psi_1),$$

$$PD = \sqrt{r^2 - x^2},$$

$$r_1 = r(\sin^2 \frac{1}{2} \theta + \cos^2 \frac{1}{4} \theta \cos^2 \phi)^{\frac{1}{2}}, \text{ area } ABC = 2rr_1 \sin \frac{1}{2} \theta \sin \psi \cos(\psi - \psi_1),$$

$$\text{volume of tetrahedron } D-ABC = \frac{1}{3} SP \cdot \text{area } ABC = \frac{2}{3} rr_1 (x + r \cos \frac{1}{2} \theta \sin \phi) \sin \frac{1}{2} \theta \times \sin \theta \cos(\psi - \psi_1).$$

Hence, we have for the required average volume, $V = \frac{4}{105} \pi r^3 = \frac{1}{35}$ of the volume of the given sphere.



[For the integration, see solution in last issue, page 113, where the figure for Professor Zerr's solution was inserted for the one belonging to Professor Walker's solution. F.]

131. Proposed by LON C. WALKER, A. M., Graduate Student, Leland Stanford Jr. University, Cal.

A sphere is described with its center within a given sphere, and its surface intersecting the surface of the given sphere. The average volume common to both spheres is $10/21$ of the volume of the given sphere.

Solution by the PROPOSER.

Let M be the center of the given sphere, and N that of the random sphere.

Put $BM=a$, $NB=x$, $MN=y$, $\angle MBN=\theta$, $\angle BMN=\phi$, $\angle BNM=\psi$.

Then we have

$$\cos\phi = \frac{1}{y}(a - x\cos\theta),$$

$$\cos\psi = \frac{1}{y}(x - a\cos\theta),$$

$$y^2 = a^2 + x^2 - 2ax\cos\theta.$$

Volume of spherical section $BACM = \frac{2}{3}\pi a^3(1 - \cos\phi) = \frac{2}{3}\pi[a^3 - 1/y(a^4 - a^3\cos\theta)]$;

volume of spherical sector $BEON = \frac{2}{3}\pi x^3(1 - \cos\psi) = \frac{2}{3}\pi[x^3 - 1/y(x^4 - ax^3\cos\theta)]$;

volume of solid $BMCN = \frac{1}{3}\pi a^2 y \sin^2\phi = \frac{1}{3}\pi(a^2 x^2/y)\sin^2\theta$;

volume of solid $BACE = S = \frac{1}{3}\pi[2a^3 + 2x^3 - 2/y(a^4 + x^4) + 2/y(a^3x + ax^3)\cos\theta - (a^2x^2/y)\sin^2\theta]$.

Hence we have for the required average volume,

$$V = \frac{\int_0^a \int_{a-x}^a S \cdot 4\pi y^2 dx dy + \int_a^{2a} \int_{x-a}^x S \cdot 4\pi y^2 dx dy}{\int_0^a \int_{a-x}^a 4\pi y^2 dx dy + \int_a^{2a} \int_{x-a}^x 4\pi y^2 dx dy}$$

$$= \frac{2\pi}{3a^4} \left\{ \int_0^a \int_{a-x}^a \left[2y^2(a^3 + x^3) - 2y(a^4 + x^4) + 2y(a^3x + ax^3)\cos\theta - a^2x^2y\sin^2\theta \right] dx dy \right.$$

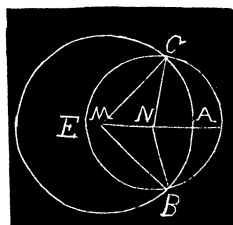
$$+ \left. \int_a^{2a} \int_{x-a}^x \left[2y^2(a^3 + x^3) - 2y(a^4 + x^4) + 2y(a^3x + ax^3)\cos\theta - a^2x^2y\sin^2\theta \right] dx dy \right\}$$

$$= \frac{2\pi}{3a^4} \left\{ \int_0^{2a} \left[\frac{2}{3}a^6 - 2a^5x + 2a^4x^2 + \frac{5}{4}a^2x^4 - 2ax^5 + \frac{1}{24}x^6 \right] dx \right.$$

$$+ \left. \int_0^a \left[\frac{2}{3}(a^3 + x^3)(x-a)^3 dx + \int_a^{2a} \frac{2}{3}(a^3 + x^3)(a-x)^3 dx \right] \right\}$$

$$= \frac{40}{3}\pi a^3 = \frac{1}{21} \text{ of the volume of the given sphere.}$$

Also solved by G. B. M. ZERR.



132. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics. The Temple College, Philadelphia, Pa.

n points are taken at random on the circumference of a given circle. Prove that the chance of the center of the circle lying within the polygon formed by joining these points is $1 - (1/2^{n-2})$.